Why some languages are nonregular ?
(The pumping lemma)

Ranveer*

A language is called a *regular language* if some deterministic finite automaton (DFA) recognizes it. If for a language there does not exist a DFA that recognizes it then the language is *nonregular*. How to prove that a language L is nonregular? Or in other words, how to prove that there does not exists a DFA that recognizes L? One trivial idea is to assume that there exists a DFA $D(L)$ which recognizes L. Now if we show that $D(L)$ also accepts strings those are not in L, then certainly L is not the language recognized by $D(L)$, hence L is nonregular. Let us see an example.

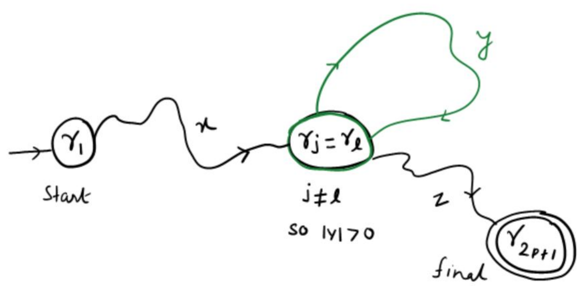

Figure 1

Example 1. *Language* $B = \{0^n1^n \mid n \geq 0\}$ *is nonregular.*

Proof. Assume that B is regular, and a DFA $D(B)$ recognises it. Let p be the number of states in $D(B)$. Consider a string $s = s_1s_2 \dots s_{2p} = 0^p1^p$. Let $R = r_1, r_2, \dots, r_{2p+1}$ be the sequence of states in $D(B)$ which process s to accept it. As $(2p+1) > p$, it implies by the pigeonhole principle that there exists a state in $D(B)$ which is repeated in R. Let this repeated state be q and its first occurrence in R be r_j and the second occurrence be r_l, see Figure 1. This implies that while processing s, the DFA $D(B)$ enters in q and then after processing some part (substring) y of s it again enters in q. Precisely for processing y the subsequence of states is $r_j, \dots, r_l$, and $y = s_j \dots s_{l-1}$. Note that y must be of nonzero length, otherwise q is not repeated. In short $q = r_j = r_l$, and $j \neq l$. Let $x = s_1 \dots s_{j-1}$ and $z = s_l \dots s_{2p}$.

To accept the string $s = xyz$, the substring x takes $D(B)$ from the starting state r_1 to r_j, y takes $D(B)$ from r_j to r_j, and z takes $D(B)$ from r_j to a final state r_{2p+1}. Let us see now what $D(B)$ will do to $xyyz$. The substring x will take $D(B)$ from the starting state r_1 to r_j, the first y takes $D(B)$ from r_j to r_j, the second y will take $D(B)$ again from r_j to r_j, and finally z takes $D(B)$ from r_j to the final state r_{2p+1}. So the string $xyyz$ is also accepted. We will consider the following three cases to show that this is impossible.

1. The substring y consists only of 0s. In this case, the string $xyyz$ will have more 0s than 1s, so $xyyz \notin B$, but it is still accepted by $D(B)$. So $D(B)$ does not recognizes B, contrary to our assumption.

2. The substring y consists only of 1s. In this case, the string $xyyz$ will have more 1s than 0s, so $xyyz \notin B$, but it is still accepted by $D(B)$. So $D(B)$ does not recognizes B, contrary to our assumption.

*Email: ranveer@iiti.ac.in

3. The substring y consists of both 0s and 1s. In this case, in $xyyz$ some 0s and 1s will be out of order, so $xyyz \notin B$, but it is still accepted by $D(B)$. So $D(B)$ does not recongnizes B, contrary to our assumption.

Thus our assumption that L is regular is wrong. This proves that L is nonregular. □

Remark: In the solution of Example 1, we see that the string xz will also be accepted by $D(B)$, why? Will this also gives a contradiction? If yes, under what condition(s) it will lead to a contradiction?

The pumping lemma

Following similar steps as in Example 1 can we prove some other language L is nonregular? By assuming the L to regular if we get some contradictions like in Example 1 then we can say L is nonregular otherwise we can not say whether L is regular or nonregular. Following the spirit of Example 1 we can now state the following lemma (theorem) which helps to decide whether some L is nonregular.

Lemma 0.1. ***Pumping lemma:*** *If A is a regular language, then there exists a number p (it is the pumping length and its depends on A) where for any string $s \in A$ of length at least p, there exists substrings x, y, z, such that $s = xyz$, and the follwing conditions are satisfied.*

1. $|y| > 0$,
2. $|xy| \leq p$, *and*
3. $xy^iz \in A$, *for* $i \geq 0$.

Proof. Let $D(A) = (Q, \Sigma, \delta, q_1, F)$ be a DFA recognising A and p be the number of states in $D(A)$. Consider a string $s = s_1 \dots s_n \in A$ of length n, where $n \geq p$. To accept s, let $D(A)$ enters in the states in sequence $R = r_1, \dots, r_{n+1}$, that is, $\delta(r_i, s_i) = r_{i+1}$ for $i = 1, \dots, n$. Among the first $p + 1$ elements in R, two must be the same state (why?..pigeonhole principle). Let the first of these two states be r_j and the second be r_l. Note that $j < l$ and $l \leq (p + 1)$.

Let $x = s_1 \dots s_{j-1}, y = s_j \dots s_{l-1}$, and $z = s_l \dots s_n$. For string $s = xyz$, the substring x takes $D(A)$ from r_1 to r_j, y takes $D(A)$ from r_j to r_j, and z takes $D(A)$ from r_j to r_{n+1}. As r_{n+1} is an accepting state, $D(A)$ must accept xy^iz, for $i \geq 0$. As $j < l$ and $l \leq (p + 1)$, the conditions $|y| > 0$ and $|xy| \leq p$ are also satisfied. □

Question 0.2. *Prove that $D = \{1^{n^2} \mid n \geq 0\}$ is nonregular.*

References

Sipser, M., 1996. Introduction to the Theory of Computation. ACM Sigact News, 27(1), pp.27-29.