Some proofs on graphs

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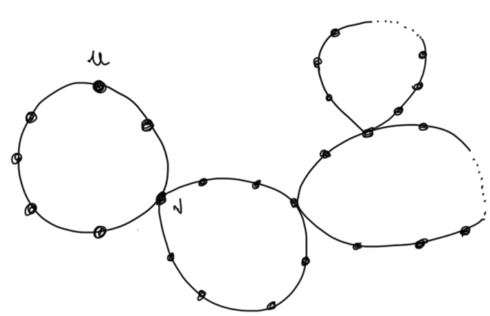
Theorem 0.1. *A connected graph is an Euler graph if and only if all the vertices are of even degree.*

Proof. Let $G = (V, E)$ be a connected graph.

1. Let G be an Euler graph. So there a closed trail T passing through the vertices and covering each edge exactly once. Suppose T starts and ends at vertex u. Let the sequence of edge in T be $(u, v_1), (v_1, v_2), \ldots, (v_k, u)$. Every time T passes through some intermediate vertex v_i it contribute two to the degree of v_i. Note that u could also be an intermediate vertex. The first and the last edge of T each contribute one to the degree of u. Hence all the vertices have even degree.

2. Let all the vertices in G have even degree. By constructing an Euler trail we will prove that G is an Euler graph. Suppose we start at a vertex u in search of closed trail T. Since each vertex has even degree we can always exit a vertex we enter, so we cannot stop at a vertex except possibly at u from where we had started. As u is also of even degree we are able to reach when we have actually find an Euler trail. If T just found covers all the edges in G, then it is an Euler graph. If not, then we remove all the edges in T from G, and get a subgraph G' formed by the remaining edges. All the vertices in G' must have even degree since both G and T have vertices with even degree. As G is connected T must touch G' at least at one vertex say v. Starting at v, we again construct a new trail T' in G'. As all the vertices of G' are of even degree, so T' terminates v. This trail T' combined with T forms a new trail trail starting and ending at u which has more edges than T (see the Figure below). This process is repeated till we obtain a closed trail that covers all the edges of G. Hence G is an Euler graph.

□



Theorem 0.2. *In a graph G the number of walks of length k between vertices i, j is equal to $A_G^k(i, j)$.*

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Proof. We prove it using induction on the length of walks. For $k = 1$, the statement is true by the definition of the adjacency matrix A_G. Suppose the statement is true for walks of length $p \geq 1$. Now consider a ij-walk w of length $p+1$. The walk w can be broken into two walks; one walk of lenth p from i to some vertex u, then another walk of length 1 from u to j, which means $u \sim j$. The total number of iu-walks of length p are $A_G^p(i,u)$. As $u \sim j$, that is, $A_G(u,j) = 1$, the total number of ij-walks length $p+1$ via u is $A_G^p(i,u)A_G(u,j)$. The total number of ij-walks of length $p+1$ via all the n vertices in G we get by $\sum_{u=1}^{n} A_G^p(i,u)A_G(u,j) = A_G^{p+1}(i,j)$. Hence the statement is true. □

Theorem 0.3. *A graph is bipartite if and only if there is no odd cycle in it.*

Proof. We will assume that graph is connected, the proof for a disconnected graph is the same, just use the same proof for different connected components.

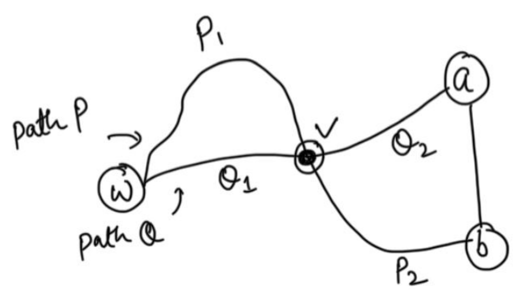
1. If a graph is bipartite. Let X, Y be the independent sets. Assume that there is an odd cycle $v_1 \sim v_2 \sim \cdots \sim v_n \sim v_1$, where, n is odd. WLOG, assume that v_1 is in X, which implies v_2 must be in Y, which further implies v_3 must be in X, and so on. That is every odd indexed vertex is in X and every even indexed vertex is in Y. Since n is odd, both v_1, v_n must be in X, but they have an edge which is impossible since X is the independent set. Hence our assumption is wrong. So all the cycles must be even.

2. If all the cycles are even. Let us choose any vertex w, and construct two sets

$$X = \{u \mid u \text{ is at odd distance from } w\},$$

$$Y = \{u \mid u \text{ is at even distance from } w\}.$$

First observe that X, Y gives a partition of the vertex set V of the graph as $X \cup Y = V$ and $X \cap Y = \phi$. We will prove that there can not be any edge between vertices of X (the same proof will tell there is no edge between vertices of Y). Suppose a, b are vertices in X with edge between them. (Look at the scenario below.)



Suppose P is a path from w to a and Q is a path from w to b (for distance we consider a shortest path). Check that both $|P|$ and $|Q|$ must be odd (in case of Y it would have been even). Next suppose v be the last vertex where P and Q intersects. Note that v may be w itself. The path from w to v along P, call it P_1 must have same length as the path from w to v along Q call it Q_1. This implies that the path from v to b call it P_2 must have same parity (either both even or both odd) as the path from v to a call it Q_2. But since there is an edge between a and b it will give a odd cycle $v - a - b - v$ which is a contradiction.

□

Theorem 0.4. *If a graph on n vertices has more than $\lfloor \frac{n^2}{4} \rfloor$ edges, then there exist a triangle in it.*

Proof. Let $G = (V, E)$ be a graph having no triangle in it. Let $\Gamma(x)$ denote the set of adjacent vertices of a vertex x. Then $\Gamma(x) \cap \Gamma(y) = \phi$ for every edge (x, y) in G, So

$$d(x) + d(y) \leq n,$$

where $d(x)$ is denote the degree of x.
Summing the above inequalilities for all the edges (x, y) in G we get

$$\sum_{x \in V(G)} d(x)^2 \leq nm, \tag{1}$$

where m is the number of edges in G. Now by Handshaking Lemma and Cauchy–Schwarz inequality, (see here) we have

$$(2m)^2 = \left(\sum_{x \in V(G)} d(x) \right)^2 \leq n \left(\sum_{x \in V(G)} d(x)^2 \right).$$

Hence, using 1,

$$(2m)^2 \leq n^2 m,$$

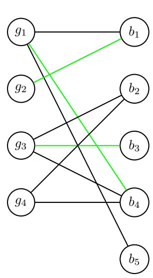
which implies $m \leq \frac{n^2}{4}$.

□

1 Matching

A matching M in G is a set of edges such that every vertex of G is incident to at most one edge in M. (In other words it is a set of vertex-disjoint edges.) The size of a matching is the number of edges in that matching. A matching is maximum when it has largest possible size.

Example 1.1. *Consider the following bipartite graph $G = (X \cup Y, E)$ of girls and boys, where $X = \{g_1, g_2, g_3, g_4\}$, $Y = \{b_1, b_2, b_3, b_4, b_5\}$. A matching M is shown in green, its size is 3.*



$M = \{(g_1, b_4), (g_2, b_1), (g_3, b_3)\}$
$|M| = 3$

is it a maximum matching?

Figure 1

A perfect matching in a graph is a matching that covers every vertex. If a perfect matching does not exists, we are interested to find a maximum matching. Let M be an arbitrary matching. If M is not maximum, how can we improve it?

1.1 Augmenting path algorithm

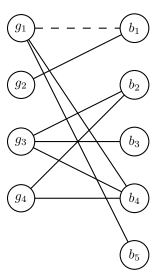
An alternating path wrt some matching M is a path that where the edges alternate in M and not in M. For example in Figure 1 an alternate path wrt to M is b_5—g_1—b_4—g_3. (If a path consists of just a single edge then it is also an alternating path.)
Finally, an augmenting path is an alternating path that starts and ends on unmatched vertex. For example in Figure 1 an augmenting path wrt to M is b_5—g_1—b_4—g_4.

Algorithm 1 Augmenting path algorithm

Input: $G = (X \cup Y, E)$
Output: A maximum matching M
Initialize: $M = \phi$;
while (there is an augmenting path P wrt M)
{
$M = (M - P) \cup (P - M)$; It is the symmetric between M and P.
}
return M;

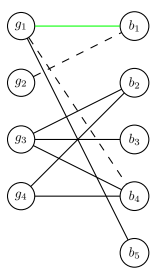
1.2 An illustration

(In an augmenting path a dashed edge denotes an edge not in matching.)



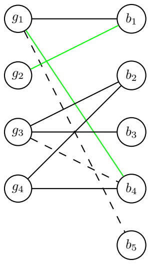
$M = \phi, \ |M| = 0$
$P = \{(g_1, b_1)\}$

1st iteration



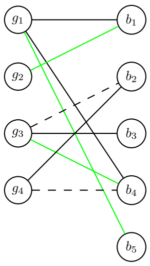
$M = (M - P) \cup (P - M) = \{(g_1, b_1)\}, \ |M| = 1$
$P = \{(b_4, g_1), (g_1, b_1), (b_1, g_2)\}$

2nd iteration



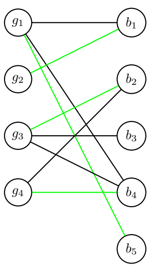
$M = \{(b_4, g_1), (b_1, g_2)\},\ |M| = 2$
$P = \{(b_5, g_1), (g_1, b_4), (b_4, g_3)\}$

3rd iteration



$M = \{(b_1, g_2), (b_5, g_1), (b_4, g_3)\},\ |M| = 3$
$P = \{(b_2, g_3), (g_3, b_4), (b_4, g_4)\}$

4th iteration



$M = \{(b_1, g_2), (b_5, g_1), (b_2, g_3), (g_4, b_4),\ |M| = 4$
Now there is no augmenting path so it is a maximum matching

Final iteration

1.3 Why the algorithm is correct

Lemma 1.2. *Every component of the symmetric difference of two matchings is a path or an even cycle.*

Example

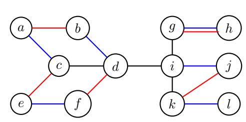


Figure 2: Two mathcing M (shown in red), and M' (shown in blue)

$M = \{(a, b), (c, e), (d, f), (g, h), (k, j)\}$
$M' = \{(a, c), (b, d), (e, f), (g, h), (i, j), (k, l)\}$
$(M - M') \cup (M' - M) = \{(a, b), (c, e), (d, f), (k, j), (a, c), (b, d), (e, f), (i, j), (k, l)\}.$
Proof of Lemma 1.2

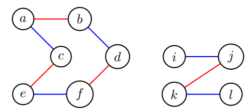


Figure 3: The subgraph on the symmetric difference of M and M'

Let M, M' be two matchings. Let $D = (M - M') \cup (M' - M)$. Check that in D a vertex has at most one incident edge from $M - M'$ and at most one incident edge from $M' - M$. Thus the maximum degree of any vertex in D is at most 2. Thus every component of symmetric difference is either a path or a cycle. Furthermore, in every path or cycle in D edges alternate between $M - M'$ and $M' - M$. Hence any cycle has to be even.

Theorem 1.3. *(Borge, 1957) A matching is maximum if and only if there is no augmenting path wrt M.*

Proof. $\Longrightarrow$ Let M be a maximum matching. Suppose there is an augmenting path P wrt M. Then

$$M' = (M - P) \cup (P - M).$$

But $|M'| = |M| + 1$, so M can not be a maximum matching, which leads to a contradiction.
$\Longleftarrow$ Suppose that there is no augmenting path wrt M and assume that M is not a maximum. Let M' be maximum matching, so $|M'| > |M|$. Let $D = (M - M') \cup (M' - M)$. Check that in D there are more edges from M' than from M. By Lemma 1.2 every component of D is either a path or an even cycle. The edges of every path and cycle in D alternate in $M - M'$ and $M' - M$. There must be an alternating path with more edges from M' than M. This path is augmenting path wrt M, which is a contradiction. □

What is the time complexity of the augmenting path algorithm?

Theorem 1.4. *A bipartite graph $G = (X \cup Y, E)$ has an X-perfect matching if and only if for every subset $S \subseteq X$ we have $|N(S)| \geq |S|$.*

Proof. $\Longrightarrow$ Let G has an X-perfect matching. Take any subset $S \subseteq X$. Every vertex u in S must have a matching edge to some vertices in Y. Since all these edges are vertex disjoint we have $|N(S)| \geq |S|$.
$\Longleftarrow$ Suppose for every subset $S \subseteq X$ we have $|N(S)| \geq |S|$. Assume that there is no X-perfect matching. Pick a maximum matching M. Let u be an unmatched vertex in X wrt to M. Let U be the set of all the vertices those are reachable from u using an alternating path (including u itself). Let $W = X \cap U$ and $Z = Y \cap U$. Every vertex in Z must have a matching edge to a vertex in W, otherwise there will an augmenting path $u - -v$ and it will violate that M is an maximum matching. Moreover u is unmatched so $|W| \geq |Z| + 1$. The next key observation is that every vertex in W must have all the adjacent vertices in Z only, again otherwise there will be an augmenting path. This implies that the neighborhood of W must be Z, that is, $N(W) = Z$. This makes $|W| \geq |Z| + 1 = |N(W)| + 1$, which is a contradiction.

□

Theorem 1.5. *(Konig's Theorem) In a bipartite graph the size of a maximum matching is the same as the size of minimum vertex cover.*

Proof. Let $G = (X \cup Y, E)$ be a bipartite graph. Let M be a maximum matching and Q be a minimum vertex cover in G. Since $|M|$ vertices must be used to cover the edges in M we have $|Q| \geq |M|$. If we can show that there exists a matching of size $|Q|$ then we are done as then $|M| \geq |Q|$ will implies that $|Q| = |M|$.
Partition Q into $R = Q \cap X$ and $T = Q \cap Y$. Let H be the subgraph induced by the vertices in $R \cup (Y - T)$ and H' be the subgraph induced by the vertices in $T \cup (X - R)$. We will prove that H has a R-perfect matching (similarly, H' has a T-perfect matching.) Since $R \cup T = Q$ is a vertex cover, G has no edge between $X - R$ and $Y - T$. Next, for any subset $S \subseteq R$ its neighborhood $N_H(S)$ in induced subgraph H satisfies $|N_H(S)| \geq |S|$, otherwise one can substitute $N_H(S)$ for S in Q to obtain a smaller vertex cover than Q. The minimality of Q implies that condition for Halls' theorem is satisfied, and H has a R-perfect matching (similarly H' has a T-perfect matching). Since H and H' are vertex disjoint, $|R| + |T| = Q$, and we have a matching of size Q. □

Theorem 1.6. *A graph having minimum degree $\delta \geq 2$ has a cycle of length at least $\delta + 1$.*

Proof. Let $v_1 \sim v_2 \sim \cdots \sim v_k$ be a maximum length path in a graph G having minimum degree δ. The vertex will have all the neighbors in this path only. Let v_δ be the farthest neighbor of v_k. The cycle $v_\delta \ldots v_k \ldots v_\delta$ will have length at least $\delta + 1$. □

Theorem 1.7. *(Euler 1758) If a connected planar graph (can have loops) has exactly n vertices, m edges and f faces, then $n + f - m = 2$.*

Proof. We will it using induction on number of vertices n. Base step: $n = 1$. If $m = 0$, then $f = 1$ so the formula is true. If $m = 1$, then $f = 2$ so the formula is true. If $m = 2$, then $f = 3$ so the formula is true. And so on...Each added loop passes through a face and cuts it into two faces, hence the formula is true for the base case.
Induction step: since G is connected, there is an edge (u, v) which is not a loop. On contracting (u, v) we obtain a planar graph G' on $n - 1$ vertices, $m - 1$ edge, and the number of faces unchanged. For G' the theorem is true due to induction hypothesis, that is, $n - 1 + f - (m - 1) = 2$, which implies $n + f - m = 2$. □

Theorem 1.8. *Let G be a connected planar graph with $n \geq 3$ vertices and m edges. Then $m \leq 3n - 6$.*

Proof. Let f be the number of faces in G. Check that each face must have a degree ≥ 3. Since $2m = \sum_{i=1}^{f} deg(f_i)$. We have

$$2m \geq 3f,$$

that is, $f \leq \frac{2m}{3}$. By Euler formula $n + f - m$=2, which implies $n + \frac{2m}{3} - m \geq 2$. Thus $m \leq 3n - 6$. □

Theorem 1.9. *Let G be a connected planar graph with n vertices, m edges and no triangle. Then $m \leq 2n - 4$.*

Proof. Let f be the number of faces in G. Check that each face must have a degree ≥ 4 as G is triangle free. Since $2m = \sum_{i=1}^{f} deg(f_i)$. We have

$$2m \geq 4f,$$

that is, $f \leq \frac{2m}{3}$. By Euler formula $n + f - m$=2, which implies $n + \frac{m}{2} - m \geq 2$. Thus $m \leq 2n - 4$. □