Introduction to Expanders and Ramanujan Graphs

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Applications

Covers broad areas of mathematics and computer science.

1. Explicit construction of robust networks
2. Error correcting codes
3. Derandomization of random algorithms
4. Quantum cryptography
5. Analysis of algorithms in computational group theory
6. Sorting networks
7. Complexity theory

Expanders

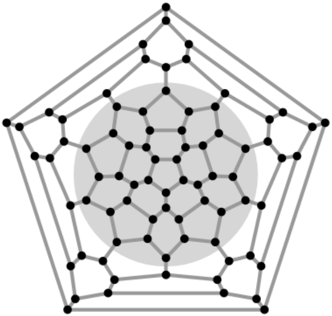
Graphs which are

1. Very sparse

2. Well-connected

Sparse

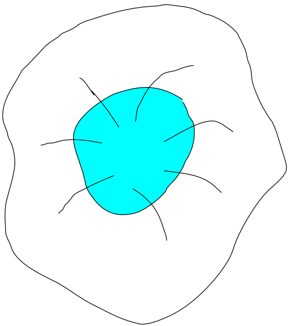
Let $G = (V, E)$ be a graph on $|V| = n$ nodes.
The number of edges $|E| << O(n^2)$.

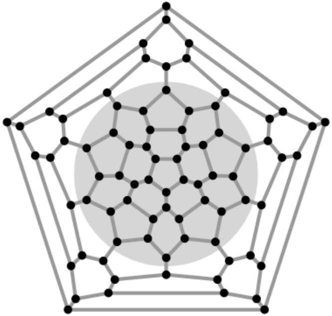


$|E| = \frac{3n}{2}$, that is, $O(n)$.

Well-connected

Every subset of the vertices has large boundary.





Brain graph

The human brain has about 10^{11} (one hundred billion) neurons. Each neuron is connected to only 7,000 other neurons on an average via synapses.

Expansion ratio

The expansion ratio of a graph $G = (V, E)$ on n vertices is

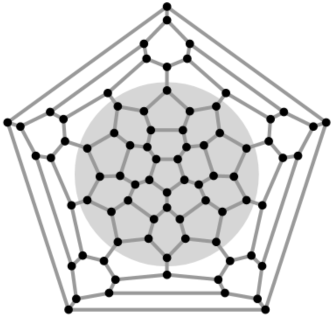
$$h(G) = \min_{S \subset V, 0 < |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|},$$

where ∂S is the boundary of S, that is, the set of edges with exactly one endpoint in S.

$h(G)$ is also known as the isoperimetric number or Cheeger constant.

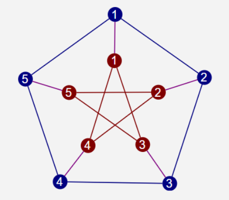
Implication

The number of edges between a subset S and its complement S' is at least $h(G) \times \min(|S|, |S'|)$.



Examples

1. Cycle C_n on n vertices: $h(C_n) \leq \frac{4}{n} \to 0$, as $n \to \infty$.
2. Complete graph K_n on n vertices: $h(K_n) \sim \frac{n}{2} \to \infty$, as $n \to \infty$.
3. Petersen graph: $h(G) = 1$.

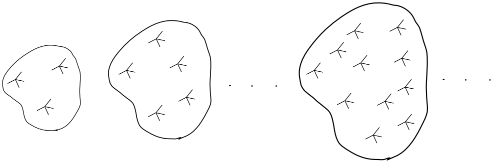


Petersen Graph

4. For connected graphs $h(G) > 0$.

Expander graphs

Definition: A family $\{G_n\}, n = 1, 2, \ldots, \infty$, of d-regular graphs and there exists $\epsilon > 0$ such that $h(G_n) \geq \epsilon$ for every n.



Family of cycle graphs (C_n) and complete graphs (K_n) are **not** expander families.

Intractable $h(G)$

No polynomial time algorithm to calculate $h(G)$.

Tomorrow if there is any polynomial time algorithm for $h(G)$, then

$$\mathbf{P{=}NP}.$$

Hence, it will settle one among the seven millennium problem of the world at present.

What to do??

Alon, Milman, 1985

Let G be a connected d-regular graph on n vertices and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix.

1. $\lambda_1 = d$.
2. $\lambda_n = -d$ iff G is bipartite graph.

Theorem[1]

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}$$

Figure: From left: Noga Alon, Milman

[1] Alon, N. and Milman, V.D., 1985. 1, isoperimetric inequalities for graphs, and superconcentrators. Journal of Combinatorial Theory, Series B, 38(1), pp.73-88.

Spectral gap

Spectral gap: $d - \lambda_2$.

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

Smaller λ_2 is better.

First explicit construction of expanders

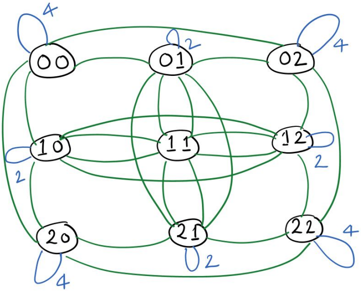
Margulis, 1973

For every natural number m, consider $G = (V, E)$, where $V = \mathbb{Z}_m \times \mathbb{Z}_m$. Every vertex (x, y) is connected to $(x \pm y, y), (x \pm (y+1), y), (x, y \pm x)$, and $(x, y \pm (x+1))$, where the arithmetic is modulo m.

Fields Medal - 1978 (Postpone due to denial of Visa to Helsinki)
Abel Prize - 2020 (Postpone due to Covid-19)

Example and analysis[2]

$\mathbb{Z}_3$

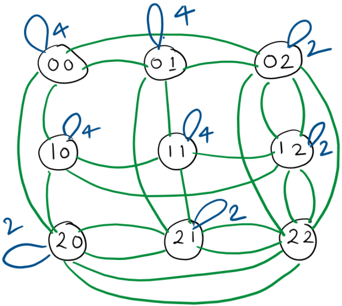


This construction yield family of 8-regular graphs with $\lambda_2 < 8$.

[2] Gabber, O. and Galil, Z., 1981. Explicit constructions of linear-sized superconcentrators. Journal of Computer and System Sciences, 22(3), pp.407-420.

A slight variant

(x, y) is connected to the vertices
$(x \pm 2y, y), (x \pm (2y+1), y), (x, y \pm 2x)$, and $(x, y \pm (2x+1))$.



This variant yields a better known bound $\lambda_2 \leq 5\sqrt{2} \sim 7.071$.

How better an expander family can be[3]

All sufficiently large d-regular graphs has

$$\lambda_2 \geq 2\sqrt{d-1} - o_n(1),$$

where $o_n(1)$, is the term tending to 0 as $n \to \infty$.

From left: Noga Alon, Ravi Bopanna

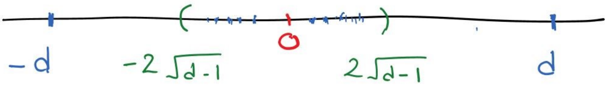
Let $\lambda = \max_{|\lambda_i|<d} |\lambda_i|, i = 1, \ldots, n$. Also, $\lambda \geq 2\sqrt{d-1} - o_n(1)$.

[3] Alon, N., 1986. Eigenvalues and expanders. Combinatorica, 6(2), pp.83-96.

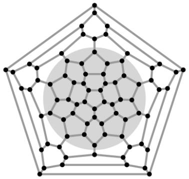
Ramanujan Graphs

The largest spectral gap

The d-regular graphs with $\lambda \leq 2\sqrt{d-1}$.



Examples



$\lambda = 2.818, h(G) = 0.25$

Other trivial examples

1. Complete graphs : $\lambda = 1$
2. Complete bipartite graphs : $\lambda = 0$
3. Petersen graph : $\lambda = 2$

Explicit construction of a family of Ramanujan graphs.

A challenge!!

The first construction

LPS[4]

Morgenstern, Lubotzky-Phillips-Sarnak: d-regular Ramanujan graphs exist when $d - 1$ is a prime power.

From left: Alex Lubotzky, Ralph S. Phillips, Peter Sarnak

It uses a Ramanujan conjecture hence they coined the name.

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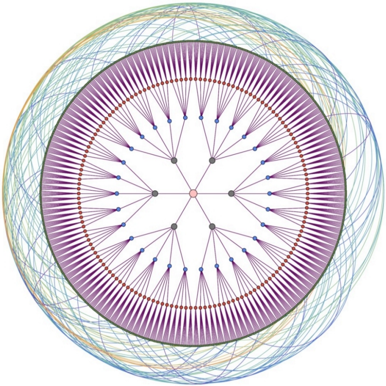
[4]Lubotzky, A., Phillips, R. and Sarnak, P., 1988. Ramanujan graphs. Combinatorica, 8(3), pp.261-277.

LPS Example

An 6-regular Ramanujan graph.



Infinite Bipartite Ramanujan graphs[5]

Srivastava-Marcus-Spielman show the existence of an infinite sequence of d-regular bipartite Ramanujan graphs.
There exist infinitely many d-regular bipartite Ramanujan graphs for any $d \geq 3$.

[5]Marcus, A.W., Spielman, D.A. and Srivastava, N., 2015. Interlacing families I: Bipartite Ramanujan graphs of all degrees. Annals of Mathematics, 182, 307–325

Random d-regular graphs

Friedman[6]

For d fixed and $\epsilon > 0$ the probability that $\lambda \leq 2\sqrt{d-1} + \epsilon$ tends to 1 as $n \to \infty$.

So a random d-regular graph is asymptotically Ramanujan.

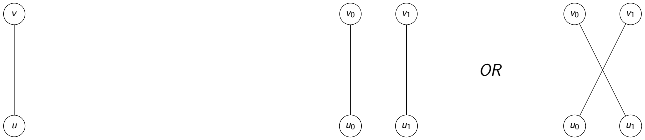
[6] Friedman, J., 2003. Relative expanders or weakly relatively Ramanujan graphs. Duke Mathematical Journal, 118(1), pp.19-35.

Given a graph $G = (V, E)$, a 2-*Lift* of G is a graph $\hat{G} = (\hat{V}, \hat{E})$ that has two vertices $\{v_0, v_1\} \subseteq \hat{V}$ for each vertex $v \in V$. If (u, v) is an edge in E, then E' can either contain the pair of edges

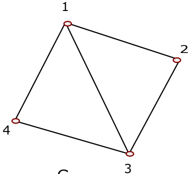
$$\{(u_0, v_0), (u_1, v_1)\},$$

or

$$\{(u_0, v_1), (u_1, v_0)\}.$$

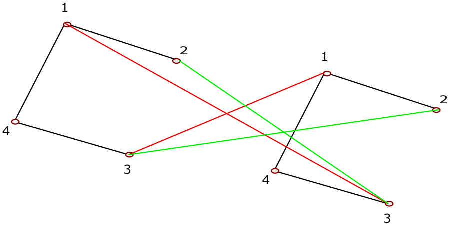


in G in $\hat{G}$ in $\hat{G}$



G

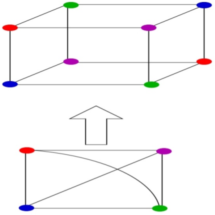
Duplicate every vertex



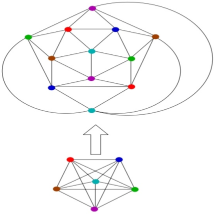
A 2-lift of G

Edges $(1, 3), (2, 3)$ are crossed in $\hat{G}$.

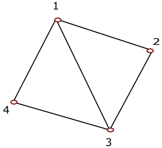
More examples

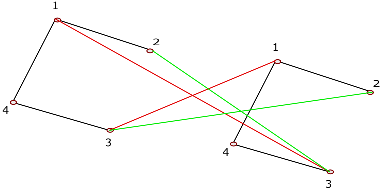


A 3-D cube is a 2-lift of K_4



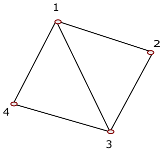
The icosahedron graph is a 2-lift of K_6

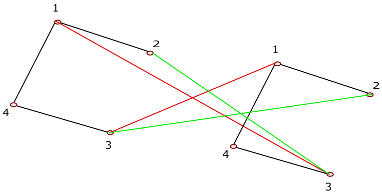




$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of A are $\{2.56, 0, -1, -1.56\}$





Signed adjacency matrix

$$A_s = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of A_s are $\{2, 1, -1, -2\}$

The eigenvalues of 2-lifts

Old eigenvalues of $\hat{G}$: $\sigma(A) = \{2.56, 0, -1, -1.56\}$.
New eigenvalues of $\hat{G}$: $\sigma(A_s) = \{2, 1, -1, -2\}$.

The eigenvalues of $\hat{G}$: $\sigma(\hat{A}) = \{2.56, 2, 1, 0, -1, -1, -1.56, -2\}$.

Theorem: $\sigma(\hat{A}) = \sigma(A) \cup \sigma(A_s)$ taken with multiplicities.

Proof

The adjacency matrix of 2-lift can be written as

$$\hat{A} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}. \tag{1}$$

Note that, $A = A_1 + A_2, A_s = A_1 - A_2$. Suppose $(\alpha, v), (\beta, u)$ be eigenpairs of A, A_s, respectively. Then

$$\left(\alpha, \begin{bmatrix} v \\ v \end{bmatrix}\right), \left(\beta, \begin{bmatrix} u \\ -u \end{bmatrix}\right)$$

are eigenpairs of $\hat{A}$. As $\begin{bmatrix} v \\ v \end{bmatrix}, \begin{bmatrix} u \\ -u \end{bmatrix}$ are orthogonal, and they are $2n$ in numbers, thus span all the eigenvectors of $\hat{A}$.

Conjecture[7]

Bilu and Linial conjectured that every d-regular graph has a signing in which all of the new eigenvalues have absolute value at most $2\sqrt{d-1}$.


From left: Nati, Bilu

For every d-regular graph there is A_s with spectral radius $O(\sqrt{d}.log^{3/2}d)$.

[7] Bilu, Y. and Linial, N., 2006. Lifts, discrepancy and nearly optimal spectral gap. Combinatorica, 26(5), pp.495-519.

Infinite Bipartite Ramanujan graphs[8]

Srivastava-Marcus-Spielman proved the conjecture for d-regular bipartite graphs.

Since the 2-lift of a bipartite graph is also bipartite, starting with a d-regular complete bipartite and inductively forming the appropriate 2-lifts gives an infinite sequence of d-regular bipartite Ramanujan graphs.

[8]Marcus, A.W., Spielman, D.A. and Srivastava, N., 2015. Interlacing families I: Bipartite Ramanujan graphs of all degrees. Annals of Mathematics, 182, 307–325

Continue..Open problem

Later Srivastava-Marcus-Spielman[9] there exist bipartite Ramanujan graphs of every degree and every number of vertices.

Michael B. Cohen[10] showed how to construct these graphs in polynomial time.

Open Problem: Are there exist infinitely many d-regular non-bipartite Ramanujan graphs for any $d \geq 3$?

[9] Marcus, A.W., Spielman, D.A. and Srivastava, N., 2018. Interlacing families IV: Bipartite Ramanujan graphs of all sizes. SIAM Journal on Computing, 47(6), pp.2488-2509.

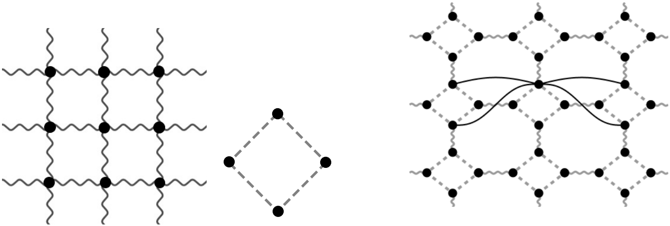
[10] Cohen, M.B., 2016, October. Ramanujan graphs in polynomial time. In 2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS) (pp. 276-281). IEEE.

Zig-zag product[11]

From left: Gold, Vadhan, Avi

[11] O. Reingold, S. Vadhan, and A. Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. Annals of Mathematics (2), 155(1):157–187, 2002.

Example



G: Grid $\mathbb{Z}^2$ (left), H: 4-cycle (middle), The replacement product $G \textcircled{r} H$ is shown with dashed edges (right), Zig-zag product $G \textcircled{z} H$ is shown with dark edges (right)

Replacement Product

G: (n, m)-graph an m-regular graph on n vertices. Assume that for each vertex there is some ordering on its m neighbors.
H: (m, d)-graph d-regular graph on m vertices.
Replacement Product $G \circledR H$:
1. Replace each vertex of G with a copy of H (call it a *cloud*). For $v \in V(G), j \in V(H)$, let (v, j) is the j-th vertex in the cloud of v.

2. Let $(u, v) \in E(G)$ be such that v is the i-th neighbor of u and u is the j-th neighbor of v. Then $((u, i), (v, j)) \in E(G \circledR H)$. Also for each $v \in V(G)$ we have $((v, i), (v, j)) \in E(G \circledR H)$, if $(i, j) \in E(H)$.

Check that $G \circledR H$ is $(d + 1)$-regular graph on nm vertices.

Zig-zag product

The Zig-zag $G \textcircled{z} H$ product is as follows:

1. The vertex set of $G \textcircled{z} H$ is the same as the vertex set of $G \textcircled{r} H$.

2. Edge $(u, i) \sim (v, j) \in E(G \textcircled{z} H)$ if there exist h and k such that $(u, i) \sim (u, h), (u, h) \sim (v, k)$ and $(v, k) \sim (v, j)$ are in $E(G \textcircled{r} H)$.

It means that vertex (v, j) can be reached from vertex (u, i) by taking a step in the cloud of u, then a step between the clouds of u and v and finally a step in the cloud of v. (Hence the name Zig-zag).

The eigenvalues of Zig-zag Product

Define a (n, d, λ)-graph as any d-regular graph on n vertices, $\lambda = \max_{|\lambda_i|<d} |\lambda_i|, i = 1, \ldots, n$.

Let G be (n, m, λ_1)-graph and H be (m, d, λ_2)-graph, then $G \textcircled{z} H$ is $(nm, d^2, f(\lambda_1, \lambda_2))$-graph, where $f(\lambda_1, \lambda_2) < \lambda_1 + \lambda_2 + \lambda_2^2$.

Other references used

1. Hoory, S., Linial, N. and Wigderson, A., 2006. Expander graphs and their applications. Bulletin of the American Mathematical Society, 43(4), pp.439-561.
2. Goldreich, O., 2011. Basic facts about expander graphs. In Studies in Complexity and Cryptography. Miscellanea on the Interplay between Randomness and Computation (pp. 451-464). Springer, Berlin, Heidelberg.
3. Sarnak, P.C., 2004. What is... an expander?. notices of the American Mathematical Society, 51(7), pp.762-763.
4. Google images