Exponential Lower Bounds for the Pfaffian Number of Graphs[1,2]

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TIFR

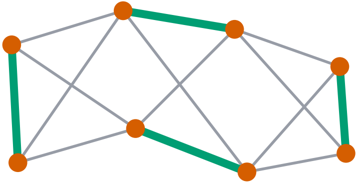
1 September 2026

[1] Joint work with Priyanshu Pant

[2] Dedicated to the memory of Prof. R. B. Bapat

What is this talk about?

1. **Counting perfect matchings in a graph**



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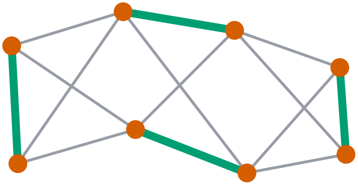
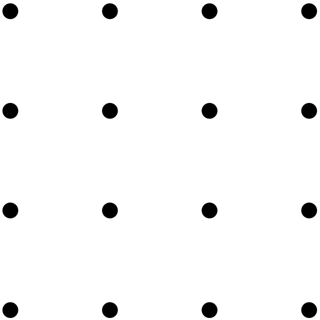


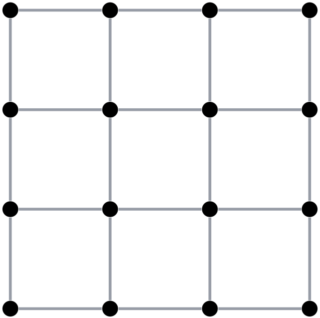
2. Permanent & Determinant

   The non-identical twins

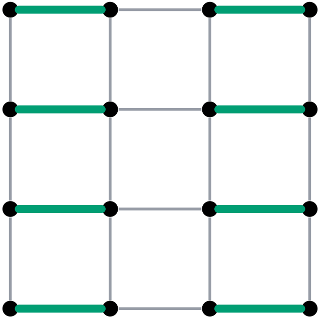
A graph has **vertices**



A graph has vertices and **edges**

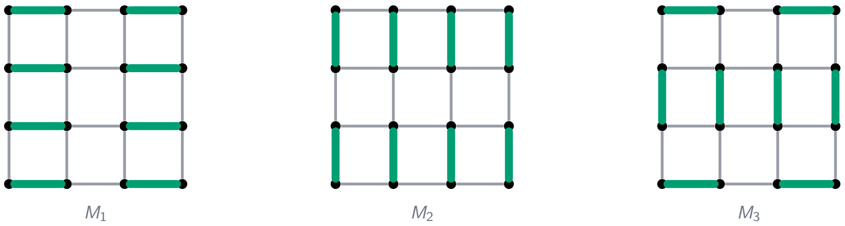


A **perfect matching**



disjoint edges covering every vertex exactly once

Graphs can have many **perfect matchings**



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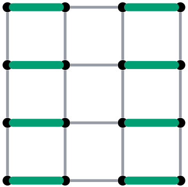
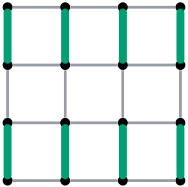
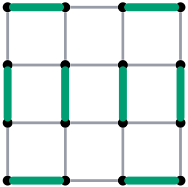


M_1



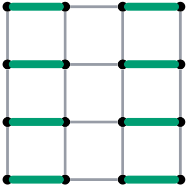
M_2



M_3

In fact, this graph has **36 perfect matchings**.

Graphs can have many **perfect matchings**



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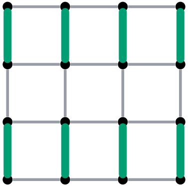

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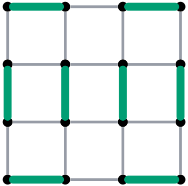

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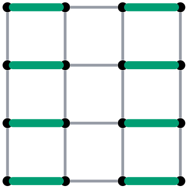
In fact, this graph has **36 perfect matchings**. **But, can we count them efficiently?**

Graphs can have many **perfect matchings**



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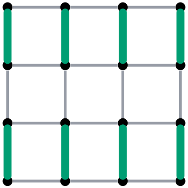

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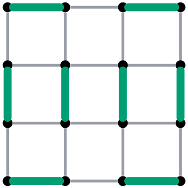


M_3

In fact, this graph has **36 perfect matchings**. **But, can we count them efficiently?**

No. In general, this is hard.

But can we count efficiently for some nice classes of graphs?

Yes.

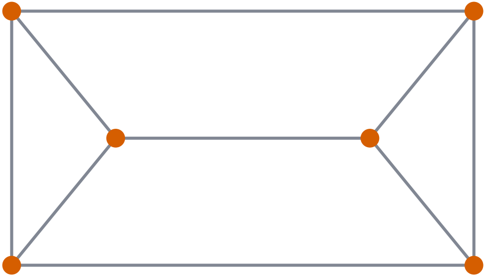
In particular, for **all planar graphs**.

More generally, whenever G admits a **Pfaffian orientation**.

Sources: M. E. Fisher, *Statistical Mechanics of Dimers on a Plane Lattice*.
P. W. Kasteleyn, *The Statistics of Dimers on a Lattice: I. The Number of Dimer Arrangements on a Quadratic Lattice*; P. W. Kasteleyn, *Dimer Statistics and Phase Transitions*.

What is a Pfaffian orientation?

What is an orientation?



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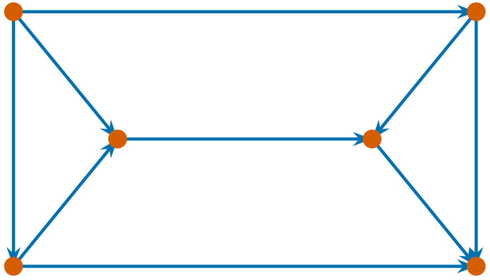
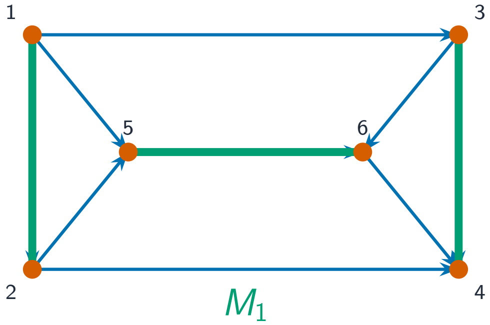


an orientation D

Given a graph G.
An **orientation** D assigns a direction to every edge of G.

What is a Pfaffian orientation?

The sign of a perfect matching



$M_1 = \{12, 34, 56\}.$

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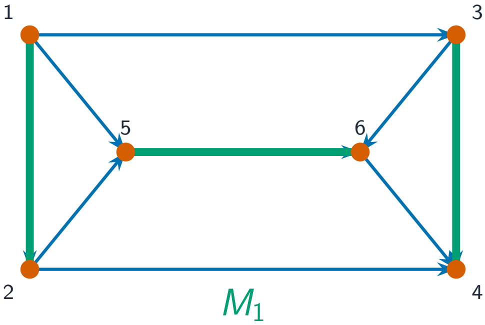

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Reading tail then head we get a permutation

$$\pi_D(M_1) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

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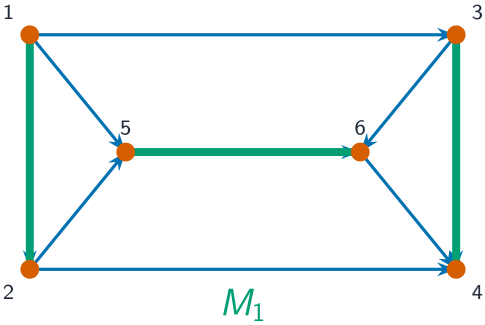

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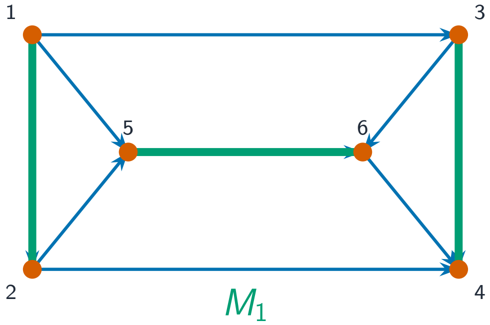

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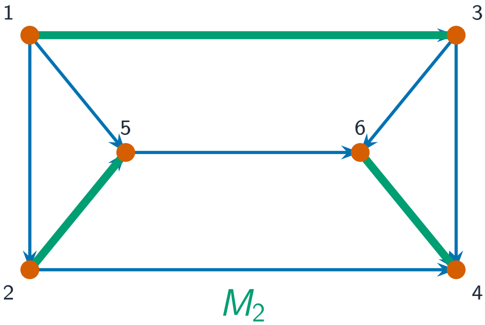

Similarly, consider

$$M_2 = \{13, 25, 46\}.$$

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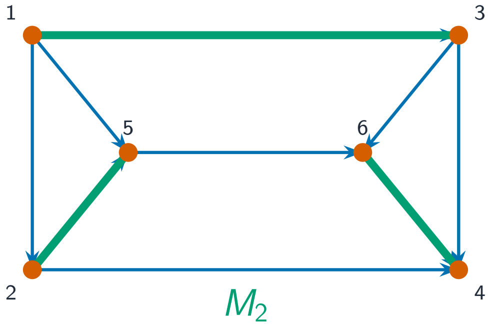

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Reading tail then head gives

$$\pi_D(M_2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 5 & 6 & 4 \end{pmatrix}.$$

$$\mathrm{sgn}_D(M_2) = -1.$$

What is a Pfaffian orientation?

The sign of a perfect matching

Formally, given some orientation D of a graph, and a perfect matching

$$M = \{u_1 v_1,\ u_2 v_2,\ \ldots,\ u_m v_m\},$$

where in D each matching edge is oriented $u_i \longrightarrow v_i$.

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and the sign of the perfect matching M with respect to D is

$$\mathrm{sign}_D(M) = \mathrm{sign}(\pi_D(M))$$

***Note** that the ordering of the edges of M does not matter.

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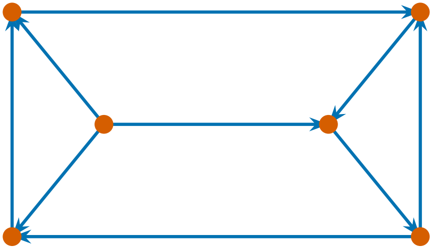

a Pfaffian orientation $D^\star$

Pfaffian orientation

An orientation in which all perfect matchings have the same sign.

$$\mathrm{sgn}_D(M) = \varepsilon \quad \forall M \in \mathcal{M}(G),$$
$$\varepsilon \in \{+1, -1\}.$$

and, G is called Pfaffian.

Why Pfaffian orientations?

The Pfaffian

For a directed graph D, assign a variable x_{ij} to each edge ij.

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The Pfaffian

For a directed graph D, assign a variable x_{ij} to each edge ij.
Define its **weighted skew-adjacency matrix** A_D by

$$(A_D)_{ij} = \begin{cases} x_{ij}, & \text{if } i \to j, \\ -x_{ij}, & \text{if } j \to i, \\ 0, & \text{otherwise.} \end{cases}$$

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Set all edge variables to 1. Then

$$|\mathrm{Pf}(A_D)| = \#\,\mathrm{PM}(G).$$

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How is this efficient?

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For any even-order skew-symmetric matrix, Cayley's identity says

$$\mathrm{Pf}(A_D)^2 = \det(A_D) \implies \mathrm{Pf}(A_D) = \sqrt{\det(A_D)}.$$

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Hence, when D is a Pfaffian orientation

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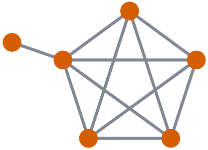
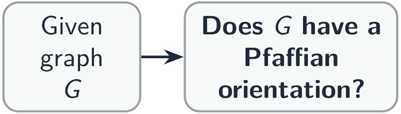
But there are **non planar Pfaffian graph** as well.



K_5 with a pendant vertex

Sources: P. W. Kasteleyn, *The Statistics of Dimers on a Lattice: I. The Number of Dimer Arrangements on a Quadratic Lattice*; P. W. Kasteleyn, *Dimer Statistics and Phase Transitions*.

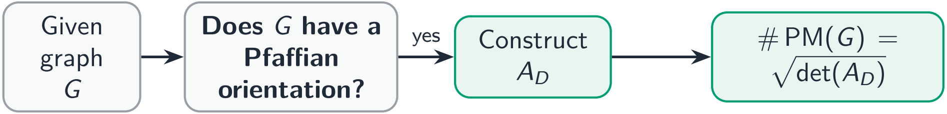
So far



So far


Given graph G
Does G have a Pfaffian orientation?
yes
Construct A_D

So far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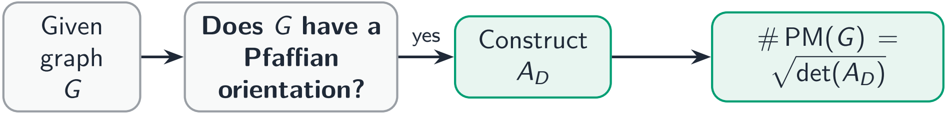
Every planar graph admits a Pfaffian orientation, found efficiently.



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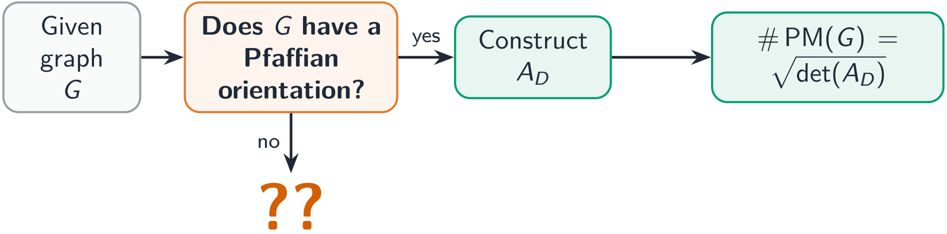

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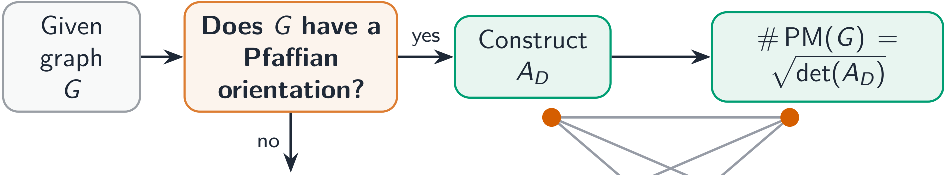

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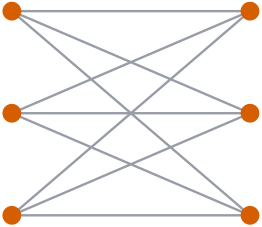

$K_{3,3}$

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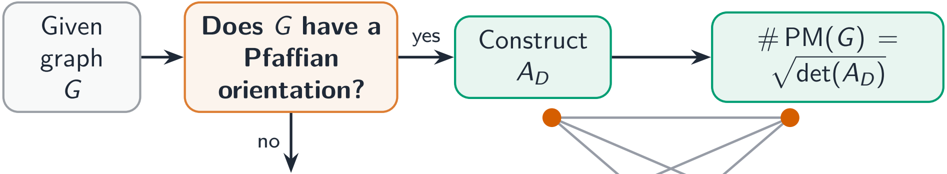

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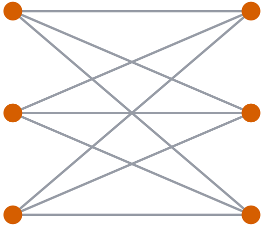

$K_{3,3}$

No polynomial-time algorithm or hardness proof known

A route for every graph

What if one Pfaffian is not enough?

The **perfect-matching polynomial** of a graph G is defined as

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But if G is has no such orientations, one can combine Pfaffians from different orientations

$$\mathrm{PM}(G) = \sum_{i=1}^{k} c_i \,\mathrm{Pf}(A_{D_i}), \quad c_i \in \mathbb{R}.$$

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A graph admitting such a representation is **k-Pfaffian**. But how large can k be for graphs in general? The minimum number of orientations needed, that is k, is called the **Pfaffian number**. That is,

$$\mathrm{pf}(G) = \min\{k : G \text{ is } k\text{-Pfaffian}\}.$$

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From planar graphs to higher genus

A graph G has orientable genus $\gamma(G) = g$ if G can be drawn without crossings on an orientable surface with g handles.

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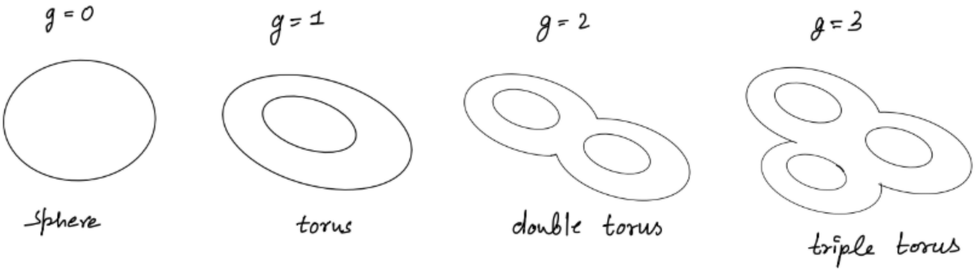
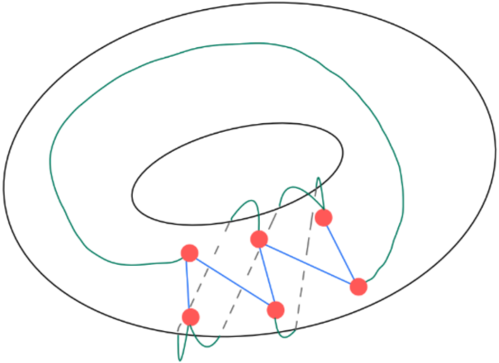


A route for every graph

From planar graphs to higher genus

A graph G has orientable genus $\gamma(G) = g$ if G can be drawn without crossings on an orientable surface with g handles.
For example, $K_{3,3}$ **has genus** 1**.**



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Our result: No.

Sources: A. Galluccio and M. Loebl, *On the Theory of Pfaffian Orientations. I. Perfect Matchings and Permanents*; G. Tesler, *Matchings in Graphs on Non-orientable Surfaces.*

Our result

Exponential lower bounds for the Pfaffian number

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For every $g \geq 1$, there is a graph G with $\gamma(G) \leq g$ such that

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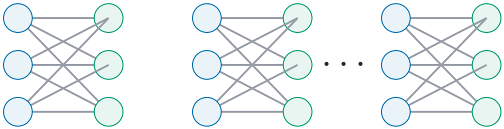

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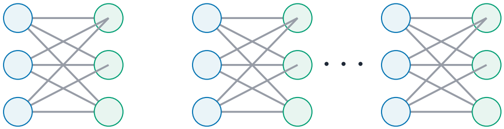

g copies of $K_{3,3}$

And, any graph containing the above graph as a spanning subgraph inherits the lower bound.

How did we prove it?

The four polynomials

Hard polynomials

Matrix permanent

$$\mathrm{per}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Perfect-matching polynomial

$$\mathrm{PM}(G) = \sum_{M \in \mathcal{M}(G)} \prod_{e \in M} x_e.$$

Easy polynomials

Matrix determinant

$$\det(A) = \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Pfaffian of an orientation D

$$\mathrm{Pf}(A_D) = \sum_{M \in \mathcal{M}(G)} \mathrm{sgn}_D(M) \prod_{e \in M} x_e.$$

How did we prove it?

The four polynomials. For Bipartite graphs: $\mathrm{PM} = \mathrm{per}$

Let $G = (U \sqcup V, E)$, with $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. Define the biadjacency matrix $B_G = (b_{ij})$ by

$$b_{ij} = \begin{cases} x_{u_i v_j}, & u_i v_j \in E, \\ 0, & u_i v_j \notin E. \end{cases}$$

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Each $\pi \in S_n$ selects

$$M_\pi = \{u_i v_{\pi(i)} : i \in [n]\}.$$

The term is nonzero exactly when M_π is a perfect matching; then it equals

$$\prod_{e \in M_\pi} x_e.$$

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Let D be an orientation of G. Its sign matrix is $S_D = (s_{ij}) \in \{\pm 1\}^{n \times n}$, where s_{ij} records the orientation of the edge $u_i v_j$.

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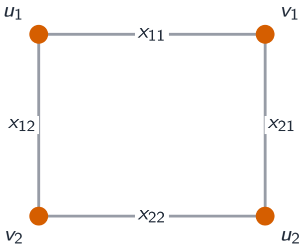
$$\mathbf{Pf}(A_D) = \epsilon_n \, \mathbf{det}(S_D \circ B_G).$$

A small example

Perfect matchings and the permanent

$$G = K_{2,2} = C_4$$



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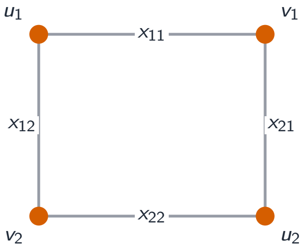

Its perfect matchings are

$$\mathcal{M}(G) = \left\{ \begin{aligned} M_1 &= \{u_1 v_1,\ u_2 v_2\}, \\ M_2 &= \{u_1 v_2,\ u_2 v_1\} \end{aligned} \right\}.$$

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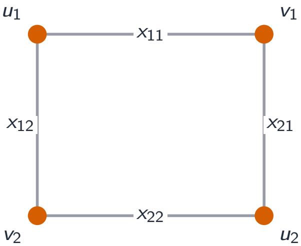

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The perfect-matching polynomial is

$$\mathrm{PM}(G) = \sum_{M \in \mathcal{M}(G)} \prod_{e \in M} x_e.$$

Hence

$$\mathrm{PM}(G) = x_{11} x_{22} + x_{12} x_{21}.$$

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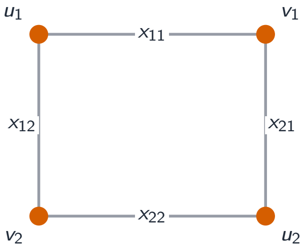

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A small example

The determinant and the Pfaffian

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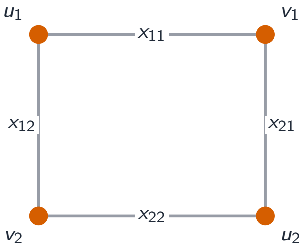

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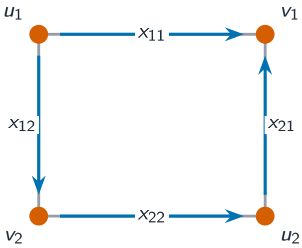

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$$S_D \circ B_G = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{22} \end{pmatrix}.$$

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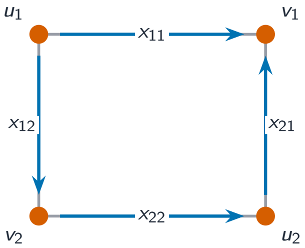

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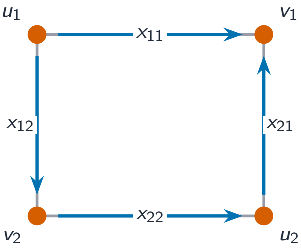

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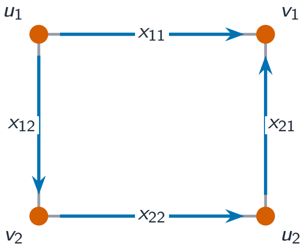

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Hence,

$$\mathrm{PM}(G) = \mathrm{per}(B_G) = \pm\,\mathrm{Pf}(A_D) = -\det(S_D \circ B_G).$$

How did we prove it?

From Pfaffians to determinants

Let G be bipartite with biadjacency matrix B_G. If G is k-Pfaffian, then

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Thus it suffices to show:

Theorem

Let A be an $n \times n$ matrix of distinct variables. If, as a polynomial identity, $\mathrm{per}(A) = \sum_{i=1}^{k} c_i \det(S_i \circ A)$ *then* $k \geq \left\lceil \left(\frac{8}{3}\right)^{\lfloor n/3 \rfloor} \right\rceil$.

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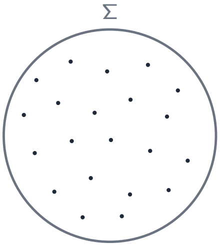
Proof.

Consider the family $\Sigma = \{\pm 1\}^{3\times 3}$, $\quad |\Sigma| = 512$.



512 sign matrices

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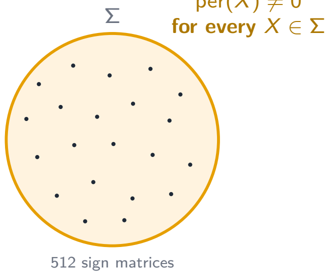
Fact 1. For every $X \in \Sigma$, $\mathrm{per}(X) \neq 0$.



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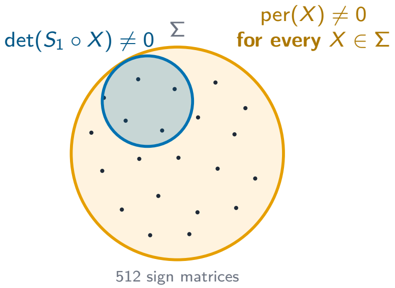

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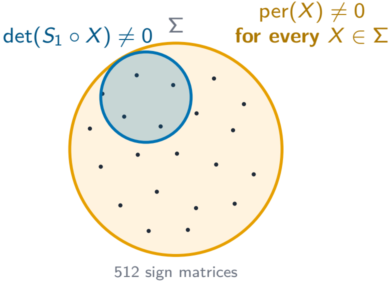

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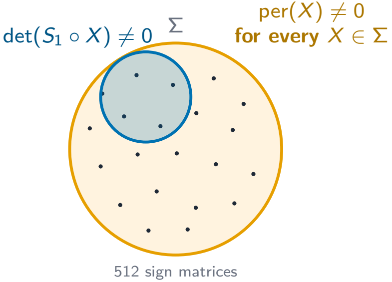
LHS $\neq 0$ for all 512 matrices $A \in \Sigma$.



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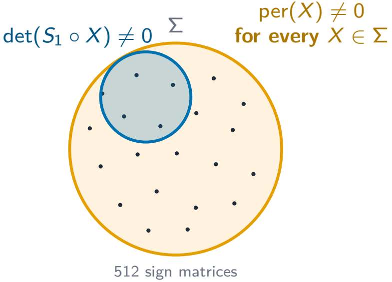
Thus, at least one term in the RHS should be nonzero for every matrix A.



How did we prove it?

The proof

Theorem

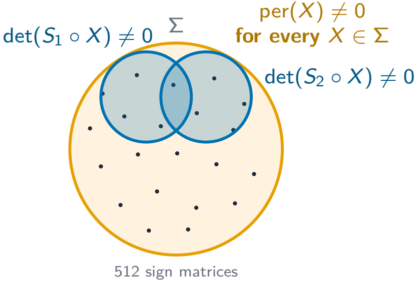
Let A be an $n \times n$ matrix of distinct variables. If, as a polynomial identity, $\mathrm{per}(A) = \sum_{i=1}^{k} c_i \det(S_i \circ A)$ *then* $k \geq \left\lceil \left(\frac{8}{3}\right)^{\lfloor n/3 \rfloor} \right\rceil$.

Proof.

Assume there exists an expression

$$\mathrm{per}(A) = \det(S_1 \circ A) + \det(S_2 \circ A) + \cdots + \det(S_k \circ A).$$

LHS $\neq 0$ for all 512 matrices $A \in \Sigma$.
Thus, at least one term in the RHS should be nonzero for every matrix A.
For a given S_i, $\det(S_i \circ A) \neq 0$ for 192 out of 512 choices $A \in \Sigma$.



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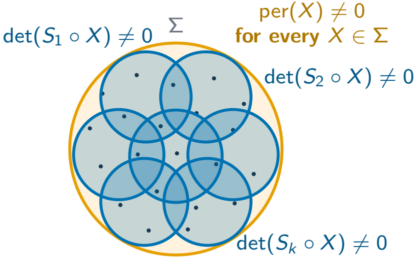

512 sign matrices

How did we prove it?

The proof

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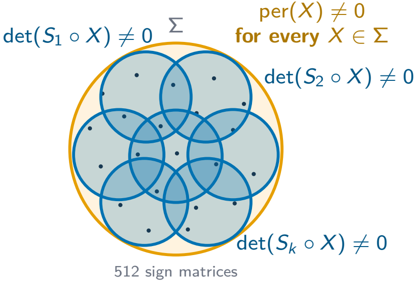
$$\mathrm{per}(A) = \det(S_1 \circ A) + \det(S_2 \circ A) + \cdots + \det(S_k \circ A).$$

LHS $\neq 0$ for all 512 matrices $A \in \Sigma$.
Thus, at least one term in the RHS should be nonzero for every matrix A.
For a given S_i, $\det(S_i \circ A) \neq 0$ for 192 out of 512 choices $A \in \Sigma$.
Since there are k such S_i, thus

$$192k \geq 512 \implies k \geq \frac{8}{3}.$$



How did we prove it?

The proof

Theorem

Let A be an $n \times n$ matrix of distinct variables. If, as a polynomial identity, $\operatorname{per}(A) = \sum_{i=1}^{k} c_i \det(S_i \circ A)$ *then* $k \geq \left\lceil \left(\frac{8}{3}\right)^{\lfloor n/3 \rfloor} \right\rceil$.

Proof.

Now we amplify. Assume

$$\operatorname{per}(A) = \det(S_1 \circ A) + \det(S_2 \circ A) + \cdots + \det(S_k \circ A)$$

Test it on $X \in \Sigma^g$.
LHS $\neq 0$ for all 512^g matrices $A \in \Sigma^g$.
But, $\det(S_1 \circ A) \neq 0$ for 192^g out of 512^g choices $A \in \Sigma^g$. Thus,

$$192^g k \geq 512^g \implies k \geq \left(\frac{8}{3}\right)^g. \quad \square$$

Let

$$X = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & X_g \end{pmatrix}.$$

where each $X_j \in \Sigma$.

How did we prove it?

Back to graphs

The weighted biadjacency matrix of

$$G_g = \underbrace{K_{3,3} \sqcup K_{3,3} \sqcup \cdots \sqcup K_{3,3}}_{g \text{ copies}}.$$

has the same block-diagonal structure $B_1 \oplus \cdots \oplus B_g$; the X_i's are the $\{\pm 1\}$-evaluations used above.

Hence our matrix lower bound gives immediately $\boxed{\mathrm{pf}(G_g) \geq \left(\frac{8}{3}\right)^g.}$

We also show:

Lemma

If** H **is a spanning subgraph of** G**, then $\mathrm{pf}(G) \geq \mathrm{pf}(H)$.

Since G_r is a spanning subgraph of $K_{3r,3r}$, $\boxed{\mathrm{pf}(K_{3r,3r}) \geq \mathrm{pf}(G_r) \geq \left(\frac{8}{3}\right)^r}$.

$$\text{Previous: } \mathrm{pf}(K_{3r,3r}) \geq 3r+1, \qquad \text{New: } \mathrm{pf}(K_{3r,3r}) \geq \left(\frac{8}{3}\right)^r.$$

Sources: E. Junchaya, A. A. Assis Miranda, and C. L. Lucchesi, *Lower Bounds for the Pfaffian Number of Graphs*.

Open directions

- What other consequences follow from the fact that representing $\mathrm{per}(A)$ by signed determinants requires exponentially many terms?
- What is the true exponential base?

$$\frac{8}{3} \leq \limsup_{g \to \infty} \mathrm{pf}_{\max}(g)^{1/g} \leq 4.$$

- What is the correct growth rate for $\mathrm{pf}(K_{n,n})$.

Thank you. Questions?