The Graphs with Exactly One Positive Eigenvalue

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Theorem 1

A connected graph G has exactly one positive eigenvalue if and only if it is a complete multipartite graph.

Proof. (⇒) Suppose G is a connected graph with exactly one positive eigenvalue. We will show that G is a complete multipartite.

First, we claim that G does not contain either P_4 (path on 4 vertices) or the graph H shown in Figure 1 as an induced subgraph. Both P_4 and H have exactly two positive eigenvalues (why? check yourself). By the Cauchy interlace theorem, if G contained either of these graphs as an induced subgraph, then G would have at least two positive eigenvalues, contradicting our assumption.

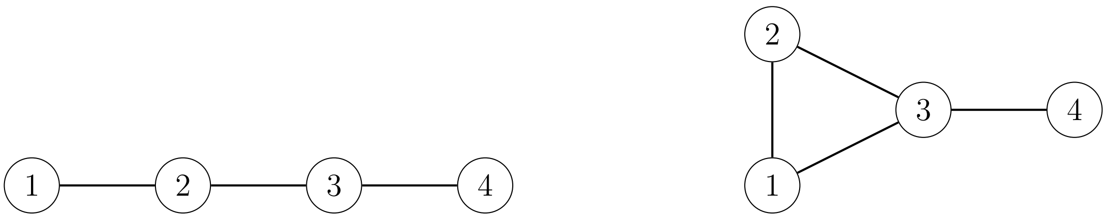

Figure 1: P_4 and H graph

Since G is P_4-free, the diameter of G is at most 2. We consider two cases:

Case 1: If the diameter of G is 1, then G is a complete graph, which is a special case of a complete multipartite graph, and we are done.

Case 2: If the diameter of G is 2, we show that any pair of non-adjacent vertices has the same neighborhood. Let u and v be two non-adjacent vertices in G. Since the diameter is 2, there exists a vertex p such that p is adjacent to both u and v.

For contradiction, suppose there exists a vertex t that is adjacent to v but not to u. Then the induced subgraph on vertices $\{u, p, v, t\}$ would be either P_4 or H, depending on whether p and t are adjacent. This contradicts our earlier claim. Therefore, any two non-adjacent vertices in G must have identical neighborhoods, which implies G is a complete multipartite.

(⇐) Now we prove that if G is a complete multipartite graph, then G has exactly one positive eigenvalue.

Let G be a complete multipartite graph with partitions $S_1, S_2, \ldots, S_p$ containing a total of n vertices, where we assume without loss of generality that $|S_1| \leq |S_2| \leq \cdots \leq |S_p|$. We proceed by induction on the number of vertices n.

Base Case: When $n = p$, we have $|S_i| = 1$ for all i, meaning G is a complete graph K_p. It is well known that the adjacency matrix of K_p has exactly one positive eigenvalue,

$p-1$, with multiplicity 1, and the remaining eigenvalues are all -1. Thus, the claim holds for the base case.

Inductive Hypothesis: Assume that every complete multipartite graph with n vertices and p partitions has exactly one positive eigenvalue.

Inductive Step: Consider a complete multipartite graph G with $n+1$ vertices and p partitions. Since $n+1 > p$, we have $|S_p| \geq 2$. Let the eigenvalues of the adjacency matrix of G be:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}$$

Remove a vertex from the partition S_p to obtain a new graph G' with n vertices and the same p partitions. By the induction hypothesis, G' has exactly one positive eigenvalue.

Let the eigenvalues of G' be:

$$\lambda_1' \geq \lambda_2' \geq \cdots \geq \lambda_n'$$

Since both G and G' are complete multipartite graphs with p partitions, the adjacency matrices of both the graphs have rank p. That is,

$$\text{rank}(A_G) = \text{rank}(A_{G'}) = p$$

We know that rank of a symmetric matrix equals to the number of its nonzero eigenvalues. Since G' has exactly one positive eigenvalue and the rank is p, we have $\lambda_1' > 0$, $\lambda_2' = \cdots = \lambda_{n+1-p}' = 0$, and $\lambda_{n+2-p}', \ldots, \lambda_n' < 0$.

By Cauchy interlace theorem, we have:

$$\lambda_1 \geq \lambda_1' \geq \lambda_2 \geq \lambda_2' \geq \cdots \geq \lambda_n' \geq \lambda_{n+1}$$

This implies that $\lambda_{n+3-p}, \ldots, \lambda_{n+1} < 0$ and $\lambda_1 > 0$. Since the rank of G is also p, we must have $\lambda_2 = \cdots = \lambda_{n+2-p} = 0$. Therefore, G has exactly one positive eigenvalue.

This proves that a connected graph G has exactly one positive eigenvalue if and only if G is a complete multipartite. □

Theorem 2

Let G be any graph. Then G has exactly one positive eigenvalue if and only if G is a complete multipartite graph with some isolated vertices.

Proof. Suppose G is a disconnected graph with k connected components $C_1, \ldots, C_k$, and assume that each component has at least one edge. Then G must have k positive eigenvalues. Therefore, if G has exactly one positive eigenvalue, then exactly one component, say C_1, contains edges, and the remaining $k-1$ components must be empty graphs (graphs without edges). By taking C_1 as an induced subgraph, it follows from Theorem 1 that C_1 is a complete multipartite graph. □