Albertson Conjecture

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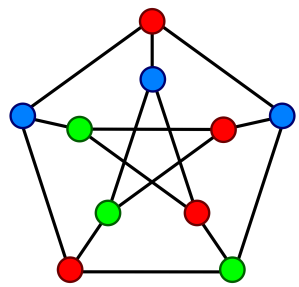
Albertson Conjecture is about a relationship between two interesting numbers, chromatic number $\chi(G)$, crossing number $\kappa(G)$ of a simple graph G (undirected, without loops, without multi-edges).

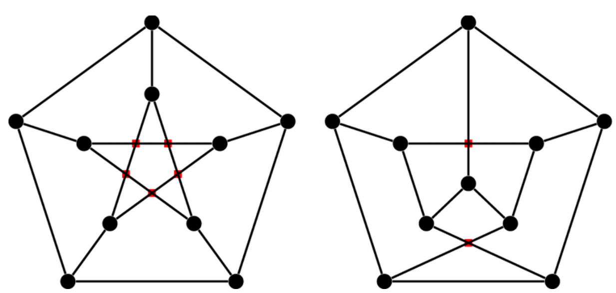
Definition 1. *The chromatic number $\chi(G)$ of a graph G is the minimum number of colors required to color the vertices so that adjacent vertices get distinct color.*

Example 1. *The chromatic number of the Petersen graph is 3.*



Definition 2. *The crossing number $\kappa(G)$ of a graph G is the lowest number of edge crossings of a plane drawing of the graph.*

Example 2. *Following two are drawing of Petersen graph. In the drawing shown on left has 5 edge crossings while the right one has 2 edge crossing. The crossing number of Petersen graph is 2.*



1 Albertson conjecture

Following is the Albertson conjecture (2007).

Conjecture 1. *Among all graphs having chromatic number n, the complete graph K_n is the one with the smallest crossing number.*

Or, equivalently

if a graph can be drawn with fewer crossings than K_n, then, according to the conjecture, it may be colored with fewer than n colors.

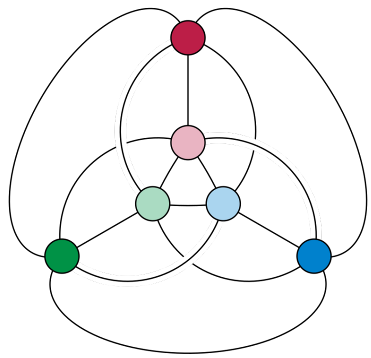
1.1 Richard K. Guy conjecture on the crossing number of K_n

Conjecture 2. *(Richard K. Guy (1972)) The crossing number of the complete graph K_n is*

$$\kappa(K_n) = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

It is known how to draw complete graphs with this many crossings, by placing the vertices in two concentric circles; what is unknown is whether there exists a better drawing with fewer crossings.

Below, the complete graph K_6 drawn with three crossings, the smallest crossing number of any graph requiring six colors.



2 A spectral bound on chromatic number

Theorem 2.1. *(Brook) For any connected graph G with maximum degree Δ_G,*

$$\chi_G \leq 1 + \Delta_G,$$

The above theorem is made tighter due to Wilf. Let $\lambda_1(G)$ be the largest eigenvalue of the adjacency matrix of a graph G.

Theorem 2.2. *(Wilf) For any connected graph G,*

$$\chi(G) \leq 1 + \lambda_1(G).$$

Proof. As $\lambda_1(G)$ is nonnegative the result is trivial if $\chi(G) = 1$. Let $\chi(G) = p \geq 2$. Let S_p be the set of all the induced subgraphs of G whose chromatic number is p. Let $H \in S_p$ be a subgraph with minimum number of vertices among all the subgraphs in S_p. That is to say, $\chi(H \setminus \{i\}) \leq p - 1$ for any vertex i of H.

We claim that the minimum degree $\delta(H) \geq p - 1$. Indeed, suppose i is a vertex of H with degree less than $p-1$. Since $\chi(H \setminus \{i\}) \leq (p-1)$, we can always properly color the vertices of $H \setminus \{i\}$ with $p-1$ colors. Since i is a vertex with degree less than $p-1$, we can also color the vertices of H properly with $p-1$ colors, a contradiction. Hence $\delta(H) \geq p - 1$. Now

$$\lambda_1(G) \geq \lambda_1(H) \geq \delta(H) \geq p - 1.$$

Hence $\chi(G) \leq 1 + \lambda_1(G)$. □

The equality in Theorem 2.2 holds if and only if G is a complete graph or a cycle graph having odd number of vertices.

For further references see Something Linky or go to the next url: `http://www.overleaf.com` or open the next file File.txt