Matrix-Tree Theorem

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In an undirected graph G a *spanning tree* is a subgraph that is connected, acyclic, and covers all the vertices of G. See graph in Figure 1(a), all its spanning trees are shown in Figure 2 (in green). Matrix-Tree Theorem gives the number of spanning-tree in an undirected graph in a polynomial time, $O(n^{2.37})$, where n is the number of vertices in G. Before we state and prove the theorem we need some preliminaries and notations.

Incidence matrix and Laplacian matrix

Let G be an undirected graph with vertex-set $V(G) = \{1, \ldots, n\}$ and edge-set $E(G) = \{e_1, \ldots, e_m\}$. To each edge of G assign an orientation (direction), which is arbitrary. The *incidence matrix* of G, denoted by $Q(G)$ (often just Q), is the $n \times m$ matrix defined as follows. The rows and the columns of Q are indexed by $1, \ldots, n$ and $1, \ldots, m$, respectively. The (i, j)-entry of Q is 0 if edge e_j is not incident on vertex i, otherwise it is 1 or -1 according as e_j originates or terminates at i, respectively. The matrix $L = QQ^T$ is called the *Laplacian matrix* of G.

Example 1: The incidence matrix and Laplacian matrix for the graph in Figure 1 (a) corresponding to a orientation of edges in Figure 1 (b) are as follows.

$$Q = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

For any undirected graph check:

1. Adding all the rows of Q gives a zero row vector. Adding all the rows (columns) of L gives a zero row (column) vector.

2. Relabelling the vertices and edges of G does not change the rank of Q, does not changes the rank, determinant of L.

3. If G has n vertices and $n - 1$ edges, then G is either a tree or it is disconnected, that is, G has more than one connected component.

(a) (b)

Figure 1: (a) A spanning tree is shown in green. (b) An arbitrary orientation and labelling of edges.

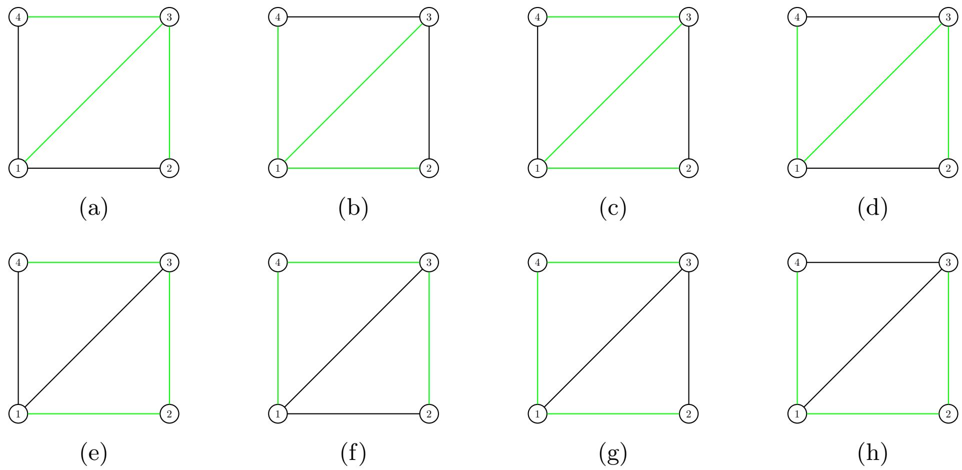

Figure 2: All the spanning trees (in green) of the graph in Figure 1 (a).

4. If G is a tree having at least two vertices, then there are at least two pendant vertices (vertices which are incident on exactly one edge).

Some more notations: Let A be an $n \times m$ matrix and $S \subset \{1, \ldots, n\}, T \subset \{1, \ldots, m\}$, the notation $A[S|T]$ denotes the sub-matrix of A whose rows and columns indices are in S, T, respectively. And the notation $A(S|T)$ denotes the sub-matrix of A whose rows and columns indices are not in S, T, respectively.

Matrix-Tree Theorem: *Any cofactor of L equals the number of spanning in G.*

In order to prove the Matrix-Tree Theorem we first need the following two lemmas and Cauchy-Binet Theorem.

Lemma 0.1: All the cofactors of L are equal.

Proof. As mentioned earlier the sum of all the rows of L gives a zero vector. Consider the sub-matrix $L(1|1)$. Its first row vector is $[L(2,2), L(2,3), \ldots, L(2,n)]$. Now consider the sub-matrix $L(2|1)$. Except its first row add all the other rows to the first one, this gives the row vector $-[L(2,2), L(2,3), \ldots, L(2,n)]$. Hence $\det(L(2|1)) = -\det(L(1|1))$. Similarly, we can prove that $\det(L(i|i)) = (-1)^{j+i} \det(L(j|i))$ for any i, j. Hence all the cofactors are the same.

□

Lemma 0.2: Let G be a connected graph on n vertices and m edges. If $T \subset \{1, \ldots, m\}$ with $|T| = n-1$, and let H be the subgraph of G consisting of edges that correspond only to the elements of T, then H forms a spanning tree of G if and only if $\det Q_G[\{1, \ldots, n-1\}|T] = \pm 1$.

Proof. First, suppose that H is not a spanning tree of G. This is possible in two cases.

Case 1: H is subgraph on less than n vertices. Suppose $|V(H)| = k \leq n-1$.

1. If vertex $n \notin V(H)$, then the submatrix $Q_G[\{1, \ldots, n-1\}|T]$ is the incidence matrix Q_H with additional $n-1-k$ zero rows. Thus the column sum of $Q_G[\{1, \ldots, n-1\}|T]$ gives a zero row, hence $\det Q_G[\{1, \ldots, n-1\}|T] = 0$.

2. If $n \in V(H)$, then there will be at least one vertex in $\{1, \ldots, n-1\}$ which is not in $V(H)$. The row corresponding to this vertex is all-zero in $Q_G[\{1, \ldots, n-1\}|T]$, hence $\det Q_G[\{1, \ldots, n-1\}|T] = 0$.

Case 2: H is an spanning subgraph on n vertices but not a tree. Note that as there are $n-1$ edges, in this case there has to be at least two components in H (observation 3). Let C be a connected component of H, and without loss of generality let $n \notin V(C)$. Without loss of generality, let $V(C) = \{1, \ldots, k\}, k < n$. The sub-matrix $Q_G[\{1, \ldots, n-1\}|T]$ can be written as

$$Q_G[\{1, \ldots, n-1\}|T] = \begin{bmatrix} Q_C & \mathbf{0} \\ \mathbf{0} & \tilde{Q}_C \end{bmatrix},$$

where Q_C is the incidence matrix of a component C, and $\tilde{Q}_C$ is the incidence matrix of subgraph induced by the edges of H not in C. In $Q_G[\{1, \ldots, n-1\}|T]$ adding the rows $2, \ldots, k$ to the first row results in the zero row. Hence, $\det Q_G[\{1, \ldots, n-1\}|T] = 0$.

Conversely, suppose H is a spanning tree of G on the edges $e_1, \ldots, e_{n-1}$. Note that, H must have at least two vertices with degree 1. Pick one such a vertex, and without loss of generality assume that e_1 is incident on this vertex, now label this vertex as 1. Next, consider the tree on $V(H) \setminus 1$ vertices, and pick a vertex with degree 1 in it, assume that e_2 is incident on this vertex, and label this vertex as 2. Continue such a labeling of vertices for all edges $e_1, \ldots, e_{n-1}$. This relabelling of vertices and edges gives a $n-1$ order square matrix which is a permutation of rows and columns of matrix $Q_G[\{1, \ldots, n-1\}|T]$. Note that this resulting matrix is a lower triangular matrix with diagonal entries ± 1. Hence, $\det Q_G[\{1, \ldots, n-1\}|T] = \pm 1$. $\square$

We are now just one step away from proving the Matrix-Tree Theorem. For the final step we need the following famous theorem known as Cauchy-Binet Theorem.

Theorem 0.3: **(Cauchy-Binet Theorem)**: Let A and B be two $n \times m$ and $m \times n$ matrices, respectively, for some positive integers n and m with $n \leq m$. Then

$$\det AB = \sum_T \det A[\{1, \ldots, n\}|T] \det B[T|\{1, \ldots, n\}],$$

where the summation runs over all subsets T of $\{1, \ldots, m\}$ with $|T| = n$.

Example 2: Consider $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix}$. $AB = \begin{bmatrix} 4 & -7 \\ 1 & -2 \end{bmatrix}$, $\det AB = -1$. By Cauchy-Binet Theorem

$$\begin{aligned} \det AB =& \det \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} \det \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} + \det \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \det \begin{bmatrix} 3 & -1 \\ -1 & -1 \end{bmatrix} \\ &+ \det \begin{bmatrix} -1 & 3 \\ -1 & 0 \end{bmatrix} \det \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} \\ =& -1. \end{aligned}$$

We are now ready to prove the Matrix-Tree Theorem.

Proof of Matrix-Tree Theorem: Without loss of generality we prove that $\det L_G(n|n)$ equals the number of spanning trees in G.

$$\begin{aligned} \det L_G(n|n) =& \det Q_G Q_G^T(n|n) \\ =& \sum_T (\det Q_G[1, \ldots, n-1|T] \det Q_G^T[T|1, \ldots, n-1]) \quad \text{(Cauchy-Binet Theorem)} \\ =& \sum_T (\det Q_G[1, \ldots, n-1|T])^2, \end{aligned}$$

where the summation runs over all the subsets T of $\{1, \ldots, m\}$, with $|T| = n-1$. By Lemma 0.2 $\det Q_G[1, \ldots, n-1|T] = \pm 1$ when the edges corresponding to T forms a spanning tree of G. This completes the proof.

For more on Matrix-Tree Theorem see [1, 2].

References

[1] R. B. Bapat, *Graphs and Matrices*. Springer, 2014.

[2] D. M. Cvetkovic, C. DM *et al.*, "Spectra of graphs. theory and application," 1980.